Two differential equations

Solve the following differential equations:

1.
$$\left(\frac{1}{x} + \frac{y}{(\cos(x))^2}\right) dx + \left(\frac{1}{y} + \tan(x)\right) dy = 0$$

2.
$$xy\,(5 + \ln(y/x))\,dx - x^2 \ln(y/x) dy = 0$$

Solution

1. It is an exact differential equation that has the form $P(x,y)dx + Q(x,y)dy = 0$ The condition that must be satisfied is $P_y = Q_x$. We check:

$$P_y = \frac{1}{[\cos(x)]^2}$$

$$Q_x = \frac{1}{[\cos(x)]^2}$$

The conditions are met, now we have to integrate both $P(x,y)$ and $Q(x,y)$:

$$\int \left(\frac{1}{x} + \frac{y}{(\cos(x))^2} \right) dx = \ln(|x|) + y\tan(x)$$

$$\int \left(\frac{1}{y} + \tan(x) \right) dy = \ln(|y|) + y\tan(x)$$

The result will be the sum of the different parts, that is: $\ln(|x|) + \ln(|y|)$ added to the part that is repeated $y\tan(x)$:

$$C = \ln(|x|) + \ln(|y|) + y\tan(x)$$

2. It is a homogeneous differential equation since:

$$P(hx, hy) = h^2 xy(5 + \ln(\frac{hy}{hx})) = h^2 P(x,y)$$

$$Q(hx, hy) = x^2 h^2 \ln(\frac{hy}{hx}) = h^2 Q(x,y)$$

I divide the differential equation by x^2:

$$\frac{xy}{x^2}\left(5 + \ln(y/x)\right) dx - \frac{x^2 \ln(y/x)}{x^2} dy = \frac{y}{x}\left(5 + \ln(y/x)\right) dx - \ln(y/x) dy$$

I make the substitution $v = y/x$ and also: $dy = vdx + xdv$

$$v(5 + \ln(v))dx - \ln(v)(vdx + xdv) = 0$$

Rearranging:

$$5vdx + \ln(v)vdx - \ln(v)vdx - x\ln(v)dv = 0$$

$$5vdx - x\ln(v)dv = 0$$

$$\frac{5}{x}dx = \frac{\ln(v)}{v}dv$$

Integrating both sides:

$$5\ln(|x|) = \frac{\ln(v)^2}{2} + C$$

Substituting back $v = y/x$

$$5\ln(|x|) = \frac{\ln(y/x)^2}{2} + C$$